Tutorial 10

April 13, 2017

1. Check the validity of the maximum principle for the harmonic function $f(x, y) = \frac{1-x^2-y^2}{1-2x+x^2+y^2}$ in the disk $\bar{D} = \{x^2 + y^2 \le 1\}$.

   Solution: The maximum principle is not valid for $f(x, y)$ in the closed unit disk. More precisely, $f(x, y) = \frac{1-x^2-y^2}{(1-x)^2+y^2}$, then $f(x, y) > 0$ on the open disk and in particular $f(0, 0) = 1$. On the other hand, $f(x, y) = 0$ on the unit circle $x^2 + y^2 = 1$ except $x = 1, y = 0$, and $\lim_{x\to 1, y\to 0} f(x, y)$ doesn't exist.

   However, this example does not voilate the maximum principle, since $f(x, y)$ isn't continuous up to the boundary.

2. **The Exterior of a Circle**

   Consider the following Dirichlet problem for the exterior of a circle

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2 + y^2 > a^2 \\ u = h(\theta), & x^2 + y^2 = a^2 \\ u \text{ is bounded as} & x^2 + y^2 \to \infty \end{cases}$$

   Solution: In polar coordinates, it suffices to solve

$$\begin{cases} r^2 u_{rr} + r u_r + u_{\theta\theta} = 0, & a < r < \infty, 0 < \theta < 2\pi \\ u = h(\theta), & 0 < \theta < 2\pi \\ u(r, 0) = u(r, 2\pi), & a < r < \infty \\ u_\theta(r, 0) = u_\theta(r, 2\pi), & a < r < \infty \\ u \text{ is bounded as} & r \to \infty \end{cases}$$

   Find a seperable solution in polar coordinates, $u = R(r)\Theta(\theta)$. Thus by the equation, we have

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda(\text{constant})$$

   Solve the eigenvalue problem

$$\begin{cases} \Theta'' = -\lambda\Theta, & 0 < \theta < 2\pi \\ \Theta(0) = \Theta(2\pi), & \Theta'(0) = \Theta'(2\pi) \end{cases}$$

   Thus the eigenvalues are $\lambda_n = n^2$ and the corresponding eigenfunctions are

$$\Theta_n = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0, 1, 2, \cdots$$

   It remains to solve

$$r^2 R'' + rR' - \lambda R = 0, \quad a < r < \infty$$

   When $n = 0$, $r^2 R'' + rR' = 0$, thus $R_0(r) = c_0 + d_0 \ln r$. When $n \ge 1$, $R_n(r) = c_n r^{-n} + d_n r^n$. Since u is bounded as $r \to \infty$, thus $R_0(r) = c_0, R_n(r) = c_n r^{-n}$

Thus

$$\begin{aligned} u(r,\theta) =& \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta) \\ =& a_0c_0 + \sum_{n=1}^{\infty} c_n r^{-n}(a_n \cos n\theta + b_n \sin n\theta) \\ =& \frac{A_0}{2} + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta) \end{aligned}$$

Set $r = a$,

$$h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$

where

$$A_n = \frac{a^n}{\pi} \int_0^{2\pi} \cos n\theta h(\theta) d\theta, \ n = 0, 1, \cdots$$

$$B_n = \frac{a^n}{\pi} \int_0^{2\pi} \sin n\theta h(\theta) d\theta, \ n = 1, 2, \cdots$$

Actually, this series can be summed explicitly.

$$u(r,\theta) = \frac{r^2 - a^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{r^2 + a^2 - 2ar\cos\theta - \phi} d\phi \quad \text{(in polar coordinates)}$$

$$u(\vec{x}) = \frac{|\vec{x}|^2 - a^2}{2\pi a} \int_{|\vec{x}|=a} \frac{u(\vec{x}')}{|\vec{x} - \vec{x}'|^2} dS(\vec{x}') \quad \text{(in rectangle coordinates)}$$

3. The annulus
   Consider the following Dirichlet problem for an annulus

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < a^2 < x^2 + y^2 < b^2 \\ u = h(\theta), & x^2 + y^2 = a^2 \\ u = g(\theta), & x^2 + y^2 = b^2 \end{cases}$$

Solution: In polar coordinates,

$$\begin{cases} r^2 u_{rr} + r u_r + u_{\theta\theta} = 0, & a < r < b, 0 < \theta < 2\pi \\ u = h(\theta), & 0 < \theta < 2\pi \\ u = g(\theta) & 0 < \theta < 2\pi \\ u(r,0) = u(r,2\pi), & a < r < b \\ u_\theta(r,0) = u_\theta(r,2\pi), & a < r < b \end{cases}$$

Find a seperable solution in polar coordinates, $u = R(r)\Theta(\theta)$,

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda(\text{constant})$$

Solve the eigenvalue problem

$$\begin{cases} \Theta'' = -\lambda\Theta, & 0 < \theta < 2\pi \\ \Theta(0) = \Theta(2\pi), & \Theta'(0) = \Theta'(2\pi) \end{cases}$$

Thus the eigenvalues are $\lambda_n = n^2$ and the corresponding eigenfunctions are

$$\Theta_n = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0, 1, 2, \cdots$$

It remains to solve

$$r^2R'' + rR' - \lambda R = 0, \quad a < r < b$$

When $n = 0$, $r^2R'' + rR' = 0$, thus $R_0(r) = c_0 + d_0 \ln r$. When $n \geq 1$, $R_n(r) = c_n r^{-n} + d_n r^n$. Thus

$$\begin{aligned} u(r, \theta) =& \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta) \\ =& a_0(c_0 + d_0 \ln r) + \sum_{n=1}^{\infty}(c_n r^{-n} + d_n r^n)(a_n \cos n\theta + b_n \sin n\theta) \\ =& \frac{A_0}{2} + \frac{B_0}{2} \ln r + \sum_{n=1}^{\infty}(A_n r^{-n} + B_n r^n) \cos n\theta + (C_n r^{-n} + D_n r^n) \sin n\theta \end{aligned}$$

Set $r = a$,

$$h(\theta) = \frac{A_0}{2} + \frac{B_0}{2} \ln a + \sum_{n=1}^{\infty}(A_n a^{-n} + B_n a^n) \cos n\theta + (C_n a^{-n} + D_n a^n) \sin n\theta$$

and $r = b$

$$g(\theta) = \frac{A_0}{2} + \frac{B_0}{2} \ln b + \sum_{n=1}^{\infty}(A_n b^{-n} + B_n b^n) \cos n\theta + (C_n b^{-n} + D_n b^n) \sin n\theta$$

where

$$\begin{aligned} \frac{A_0}{2} + \frac{B_0}{2} \ln a &= \frac{1}{\pi} \int_0^{2\pi} \cos\theta h(\theta) d\theta \\ A_n a^{-n} + B_n a^n &= \frac{1}{\pi} \int_0^{2\pi} \cos n\theta h(\theta) d\theta, \ n = 1, 2, \cdots \\ C_n a^{-n} + D_n a^n &= \frac{1}{\pi} \int_0^{2\pi} \sin n\theta h(\theta) d\theta, \ n = 1, 2, \cdots \\ \frac{A_0}{2} + \frac{B_0}{2} \ln b &= \frac{1}{\pi} \int_0^{2\pi} \cos\theta g(\theta) d\theta \\ A_n b^{-n} + B_n b^n &= \frac{1}{\pi} \int_0^{2\pi} \cos n\theta g(\theta) d\theta, \ n = 1, 2, \cdots \\ C_n b^{-n} + D_n b^n &= \frac{1}{\pi} \int_0^{2\pi} \sin n\theta g(\theta) d\theta, \ n = 1, 2, \cdots \end{aligned}$$